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Global bifurcation structure of a limiting system to the SKT competition model with cross-diffusion *

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1 Introduction

This is a joint work with Yuan Lou (The Ohio State University), Wei-Ming Ni (The Chinese University of Hong Kong and University of Minnesota), Tatsuki Mori (Osaka University), and Shota Yamakawa (Ryukoku University).

We have been interested in the cross-diffusion system

$$(P) \begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v) & \text{in } \Omega \times (0, \infty), \quad (1.1) \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2u - c_2v) & \text{in } \Omega \times (0, \infty), \quad (1.2) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \quad (1.3) \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega, \quad (1.4) \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$.

This mathematical model was proposed by Shigesada, Kawasaki and Teramoto [8] in 1979 to investigate segregation phenomena of two competing species with each other in the same habitat area. Here, $u = u(x, t)$ and $v = v(x, t)$ represent the densities of two competing species, d_1 and d_2 are their diffusion coefficients, a_1 and a_2 denote the intrinsic growth rates of these two species, b_1 and c_2 account

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for intra-specific competitions while b_2 and c_1 account for inter-specific competitions. The constants α_{11} and α_{22} represent intra-specific population pressures, also known as self-diffusion rates, and α_{12} and α_{21} are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

The effect of cross-diffusion affects the population pressure between two different kinds. It is an interesting problem to see whether this effect may give rise to a spatial segregation or not, and clarify its mechanism.

We should remark that it is well known that the important quantities involving the constants a_i, b_i, c_i ($i = 1, 2$) are only

$$A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}. \quad (1.5)$$

It seems natural to consider the following two cases separately: the "strong competition" case $B < C$ and the "weak competition" case $C < B$. The behavior of solution in case $B < C$ is very different from $C > B$.

We refer to [7] and [8] for further details of this model.

A lot of research works are done by the singular perturbation method, which started from a theoretical research by Mimura [5]. Kan-on [1] obtained some criteria on the stability of those non-constant solutions of (P). However, it is not easy to clarify the global structure of stationary solutions and stability of stationary solutions.

Lou and Ni [2], [3] started to investigate N-dimensional case and general diffusion coefficients. To investigate the cross-diffusion effects, let us put $\alpha_{11} = \alpha_{21} = \alpha_{22} = 0$ and $r := \alpha_{12}/d_1$. We have

$$(TP_r^N) \begin{cases} u_t = d_1 \Delta[(1 + rv)u] + u(a_1 - b_1u - c_1v) & \text{in } \Omega \times (0, \infty), & (1.6) \\ v_t = d_2 \Delta v + v(a_2 - b_2u - c_2v) & \text{in } \Omega \times (0, \infty), & (1.7) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), & (1.8) \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega, & (1.9) \end{cases}$$

where $u = u(x, t)$ and $v = v(x, t)$. Then, the stationary problem of (TP_r^N) is

$$(S_r^N) \begin{cases} d_1 \Delta[(1 + rv)u] + u(a_1 - b_1u - c_1v) = 0 & \text{in } \Omega, & (1.10) \\ d_2 \Delta v + v(a_2 - b_2u - c_2v) = 0 & \text{in } \Omega, & (1.11) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, & (1.12) \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega, & (1.13) \end{cases}$$

where $u = u(x)$ and $v = v(x)$.

They obtained limiting systems as $r \rightarrow \infty$ for (TP_r^N) and (S_r^N) . One of limiting systems as $r \rightarrow \infty$ are as follows. The time-dependent limiting system is

$$(\text{TP}_\infty^N) \begin{cases} \frac{\partial}{\partial t} \int_\Omega \frac{\tau}{v} dx = \int_\Omega \frac{\tau}{v} \left(a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx & \text{in } (0, \infty), \\ \frac{\partial v}{\partial t} = d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(0, t) = v_0(x) > 0 & \text{in } \Omega, \end{cases} \quad (1.14)$$

$$(1.15)$$

$$(1.16)$$

$$(1.17)$$

where $v = v(x, t)$ and $\tau = \tau(t)$ are unknown positive functions, and $\tau(t)/v(x, t)$ corresponds to $u(x, t)$. The stationary limiting system is

$$(S_\infty^N) \begin{cases} \int_\Omega \frac{\tau}{v} \left(a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, \\ d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ v(x) > 0, & \text{in } \Omega, \end{cases} \quad (1.18)$$

$$(1.19)$$

$$(1.20)$$

$$(1.21)$$

where $v = v(x)$ is an unknown positive function, τ is an unknown positive constant.

For one-dimension $\Omega := (0, 1)$, the limiting system corresponding (TP_∞^N) and (S_∞^N) are

$$(\text{TP}_\infty^1) \begin{cases} \frac{\partial}{\partial t} \left(\int_0^1 \frac{\tau}{v} dx \right) = \int_0^1 \frac{\tau}{v} \left(a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx & \text{in } (0, 1) \times (0, \infty), \\ \frac{\partial v}{\partial t} = d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau & \text{in } (0, 1), \\ v_x(0, t) = 0, \quad v_x(1, t) = 0, & \text{in } (0, \infty), \\ v(x, 0) = v_0(x) > 0, & \text{in } (0, 1), \end{cases} \quad (1.22)$$

$$(1.23)$$

$$(1.24)$$

$$(1.25)$$

and

$$(S_{\infty, \text{general}}^1) \begin{cases} \int_0^1 \frac{\tau}{v} \left(a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, \\ d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau = 0 & \text{in } (0, 1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v(x) > 0 & \text{in } (0, 1). \end{cases} \quad (1.26)$$

$$(1.27)$$

$$(1.28)$$

$$(1.29)$$

Lou, Ni and Yotsutani [4] obtained existence and non-existence of non-constant steady state solutions, the asymptotic shape of solutions, and almost clarified the structure of solutions of $(S_{\infty, \text{general}}^1)$.

In what follows, we concentrate on the monotone increasing case $v_x(x) > 0$ to understand the essence of structure of $(S_{\infty, \text{general}}^1)$.

Now, we introduce a (S_{∞}^1) as follows:

$$(S_{\infty}^1) \begin{cases} \int_0^1 \frac{\tau}{v} \left(a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, & (1.30) \\ d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau = 0 & \text{in } (0, 1), & (1.31) \\ v_x(0) = 0, \quad v_x(1) = 0, & (1.32) \\ v(x) > 0, \quad v_x(x) > 0 & \text{in } (0, 1). & (1.33) \end{cases}$$

2 Results

We first explain results in [4] for (S_{∞}^1) . As for the existence and non-existence, the following theorems are obtained:

Theorem A (Existence, weak competition). *Suppose that $C \leq B$.*

- (i) *If $B \leq A$ then there exists a solution (v, τ) of (S_{∞}^1) .*
- (ii) *If $(B + 3C)/4 < A < B$, then there exists a solution of (S_{∞}^1) . for $d_2 \in (0, \frac{2A - (B+C)}{B-C} \cdot \frac{a_2}{\pi^2})$.*

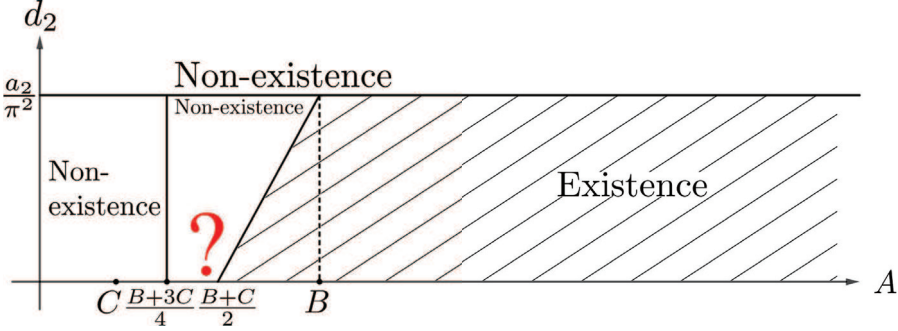


Figure 1: Existence and non-existence of solutions of (S_{∞}^1) for $C \leq B$.

Theorem B (Non-Existence, weak competition). *Suppose that $C \leq B$.*

- (i) *If $d_2 > a_2/\pi^2$, then there exists no solution of (S_{∞}^1) .*
- (ii) *If $(B + 3C)/4 < A < B$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (S_{∞}^1) for $d_2 \in (d_2^*, a_2/\pi^2)$.*
- (iii) *If $A \leq (B + 3C)/4$, there exists no solution of (S_{∞}^1) .*

Figure 1 shows the existence and non-existence region of solutions of (S_∞^1) in the case $C \leq B$ assured by theorems A and B. Here, horizontal axis is A , vertical axis is d_2 . For the case d_2 sufficiently close to 0 and $(B+3C)/4 < A < (B+C)/2$, existence and non-existence of solutions of (S_∞^1) are not clear.

Figure 2 shows the existence and non-existence region of solutions of (S_∞^1) in the case $B < C$ assured by theorems C and D. For the case $0 < d_2 < ((B+C-2A)/(C-B)) \cdot (a_2/\pi^2)$ and $B < A < (B+C)/2$, existence and non-existence of solutions of (S_∞^1) also are not clear.

Theorem C (Existence, strong competition). *Suppose that $B < C$. If*

$$\max \left\{ 0, \frac{B+C-2A}{C-B} \cdot \frac{a_2}{\pi^2} \right\} < d_2 < \frac{a_2}{\pi^2}, \quad (2.1)$$

then there exists a solution (v, τ) of (S_∞^1) .

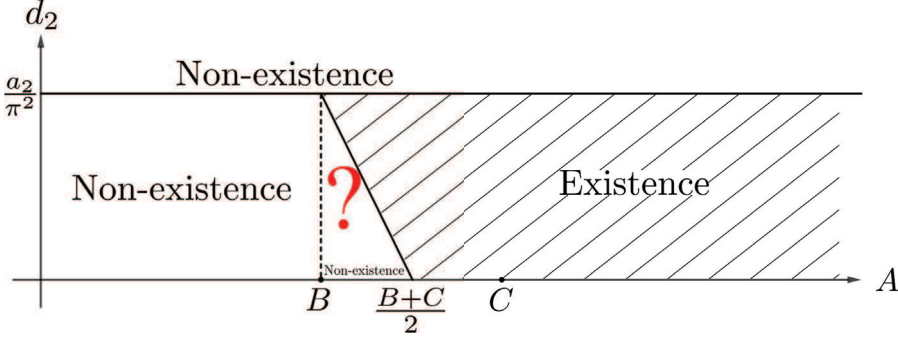
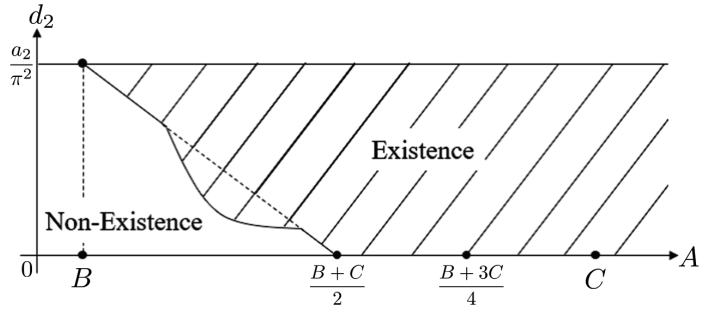
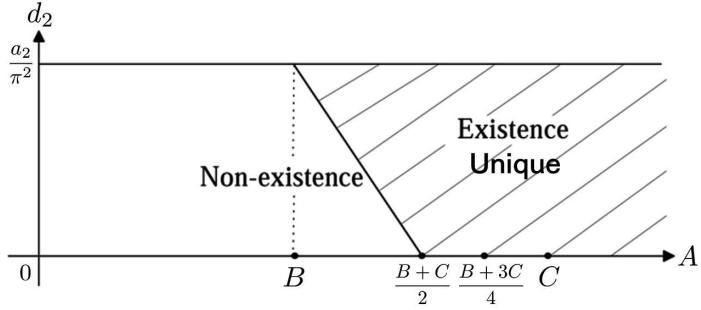
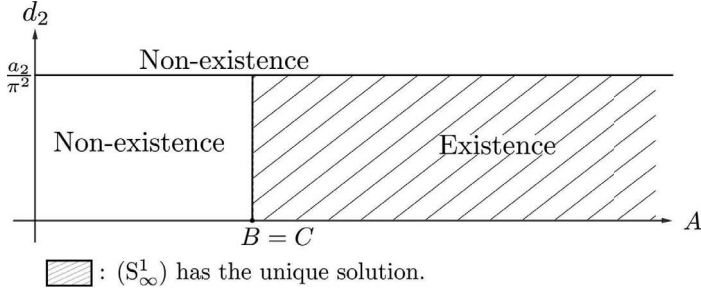


Figure 2: Existence and non-existence of solutions of (S_∞^1) for $B < C$.

Theorem D (Non-Existence, strong competition). *Suppose that $B < C$.*

- (i) *If $d_2 > a_2/\pi^2$, then there exists no solution of (S_∞^1) .*
- (ii) *If $B \leq A < (B+C)/2$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (S_∞^1) for $d_2 \in (0, d_2^*]$.*
- (iii) *If $A < B$, there exists no solution of (S_∞^1) .*

In [9], Lou, Ni and Yotsutani conjectured that the situation of existence, non-existence and the uniqueness drastically changes at $C = (7/3)B$. For the case $B < C \leq (7/3)B$, the uniqueness seems to hold as shown in Figures 3 and 4. Recently, we have found a mathematical proof of this case.



On the other hand, for the case $C > (7/3)B$, the existence region becomes wider as shown in Figure 5. In [6], Mori, Suzuki and Yotsutani have obtained precise numerical results with the stability and instability for this case

As explained above, existence, non-existence and multiplicity of solutions for the case $B \leq C$ are precisely understood.

In what follows in (S_∞^1) , we will concentrate on the case

$$b_1 = 1 \quad \text{and} \quad a_2 = b_2 = c_2 = 1. \quad (3.4)$$

In fact, we get from (S_∞^1) .

$$\begin{cases} \int_0^1 \frac{1}{\bar{v}} \left(\frac{A}{B} - \frac{\bar{\tau}}{\bar{v}} - \frac{C}{B} \bar{v} \right) dx = 0, \end{cases} \quad (3.5)$$

$$\begin{cases} \bar{d}_2 \bar{v}_{xx} + \bar{v}(1 - \bar{v}) - \bar{\tau} = 0 \end{cases} \quad \text{in } (0, 1), \quad (3.6)$$

$$\begin{cases} \bar{v}_x(0) = 0, \quad \bar{v}_x(1) = 0, \end{cases} \quad (3.7)$$

$$\begin{cases} \bar{v}(x) > 0, \quad \bar{v}_x(x) > 0 \end{cases} \quad \text{in } (0, 1) \quad (3.8)$$

by employing the following change of variables

$$\bar{v} := \frac{c_2}{a_2} \cdot v, \quad \bar{\tau} := \frac{b_2 c_2}{a_2^2} \cdot \tau, \quad \bar{d}_2 := \frac{d_2}{a_2}. \quad (3.9)$$

Thus, without lose of generality, we may consider the case $b_1 = 1$ and $a_2 = b_2 = c_2 = 1$.

Now, we introduce an auxiliary problem to investigate (S_∞^1) with $b_1 = a_2 = b_2 = c_2 = 1$. Let $d_2 > 0$ be given. Unknowns are a function $v = v(x)$ and a constant $\tau > 0$.

$$(E) \begin{cases} d_2 v_{xx} + v(1 - v) - \tau = 0 \end{cases} \quad \text{in } (0, 1), \quad (3.10)$$

$$\begin{cases} v(x) > 0 \text{ in } [0, 1] \text{ and } v_x(x) > 0 \end{cases} \quad \text{in } (0, 1), \quad (3.11)$$

$$\begin{cases} v_x(0) = 0, \quad v_x(1) = 0 \text{ and } \tau > 0. \end{cases} \quad (3.12)$$

Exact solutions of (E) are given in the following proposition.

Proposition 3.1. *(E) has a solution if and only if $d_2 \in (0, 1/\pi^2)$. All solutions $(v(x), \tau)$ of (E) are represented by*

$$v(x; d_2, h) = \alpha + (\beta - \alpha) \operatorname{sn}^2(K(\sqrt{h})x, \sqrt{h}), \quad (3.13)$$

$$\tau(d_2, h) = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{3} = \frac{1}{4} - 4d_2^2(h^2 - h + 1)K(\sqrt{h})^4, \quad (3.14)$$

where

$$\alpha = \frac{1}{2} - 2d_2 K(\sqrt{h})^2(h + 1), \quad (3.15)$$

$$\beta = \frac{1}{2} + 2d_2 K(\sqrt{h})^2(2h - 1), \quad (3.16)$$

$$\gamma = \frac{1}{2} + 2d_2 K(\sqrt{h})^2(2 - h). \quad (3.17)$$

Here \tilde{h} is the unique solution of an equation

$$(h+1)K(\sqrt{h})^2 = \frac{1}{4d_2} \quad (3.18)$$

in h , $K(\sqrt{h})$ is the complete elliptic integral of the 1st kind, and $\text{sn}(\cdot, \cdot)$ is Jacobi's elliptic function.

Now, we note that (1.30) with $b_1 = 1$ is rewritten as

$$\frac{\tau \int_0^1 \frac{1}{v^2} dx + c_1}{\int_0^1 \frac{1}{v} dx} = a_1. \quad (3.19)$$

Thus, let us define a function $\tilde{a}_1(h; d_2, c_1)$ by

$$\tilde{a}_1(h; d_2, c_1) := \frac{\tau \int_0^1 \frac{1}{v(x; d_2, h)^2} dx + c_1}{\int_0^1 \frac{1}{v(x; d_2, h)} dx}. \quad (3.20)$$

$\tilde{a}_1(h; d_2, c_1)$ is explicitly given in the following proposition.

Proposition 3.2. *Let $d_2 \in (0, 1/\pi^2)$, $h \in (0, \tilde{h}(d_2))$. It holds that*

$$\begin{aligned} & \tilde{a}_1(h; d_2, c_1) \\ &= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{6\alpha\beta\gamma\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right)} \\ & \quad \cdot \left((\gamma - \alpha)\alpha E(\sqrt{h}) - \alpha\gamma K(\sqrt{h}) + (\alpha\beta + \beta\gamma + \gamma\alpha)\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right) \right) \\ & \quad + \frac{\alpha K(\sqrt{h})c_1}{\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right)}, \end{aligned} \quad (3.21)$$

where α , β and γ are defined by (3.15), (3.16) and (3.17) respectively. Here, $K(\cdot)$, $E(\cdot)$ and $\Pi(\cdot, \cdot)$ are the complete elliptic integral of the 1st, 2nd and 3rd kind, respectively.

We explain the reason that the existence and non-existence regions change at $c_1 = 7/3$ ($C/B = 7/3$). We obtain

$$\tilde{a}_1(h; d_2, c_1) = \frac{1}{2} (d_2 \pi^2 (1 - c_1) + (1 + c_1)) + \tilde{a}_{1,2} \cdot h^2 + \cdots, \quad (3.22)$$

by Taylor's expansion of (3.21) in h , where

$$\tilde{a}_{1,2} := \frac{3d_2\pi^2}{64(1 - \pi^2 d_2)^2} \left((35 + 13c_1)\pi^4 d_2^2 - 14\pi^2(c_1 - 1)d_2 + (c_1 - 1) \right). \quad (3.23)$$

We check the sign of the coefficient $\tilde{a}_{1,2}$. We get $d_2 = d_+$ and d_- by solving

$$(35 + 13c_1)\pi^4 d_2^2 - 14\pi^2(c_1 - 1)d_2 + (c_1 - 1) = 0, \quad (3.24)$$

where

$$d_+ := \frac{7(c_1 - 1) + 2\sqrt{3(c_1 - 1)(3c_1 - 7)}}{\pi^2(35 + 13c_1)} \quad (3.25)$$

and

$$d_- := \frac{7(c_1 - 1) - 2\sqrt{3(c_1 - 1)(3c_1 - 7)}}{\pi^2(35 + 13c_1)}. \quad (3.26)$$

Thus,

$$\tilde{a}_{1,2} < 0 \quad \text{for} \quad 0 < c_1 < 1, \quad 0 < d_2 < d_+, \quad (3.27)$$

$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad 1 \leq c_1 \leq 7/3, \quad 0 < d_2 < 1/\pi^2, \quad (3.28)$$

$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad c_1 > 7/3, \quad d_+ \leq d_2 < 1/\pi^2, \quad (3.29)$$

$$\tilde{a}_{1,2} < 0 \quad \text{for} \quad c_1 > 7/3, \quad d_- < d_2 < d_+, \quad (3.30)$$

$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad c_1 > 7/3, \quad 0 < d_2 \leq d_-. \quad (3.31)$$

Therefore, the behavior of $\tilde{a}_1(h, d_2, c_1)$ near $h = 0$ is drastically change at $c_1 = 1$ and $c_1 = 7/3$.

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